TRINITY COLLEGE

Mathematics Specialist Units 3,4
Test 3 2019

Calculator Assumed

Vector Calculus

STUDENT'S NAME SOLUTIONS

DATE: Wednesday 15th May **TIME:** 55 minutes **MARKS**: 54

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (12 marks)

Consider the following system of equations:

$$\begin{aligned} x+\ y+\ \ \ z&=2\\ x+2y+\ (k-5)z&=2\\ 3x+2y+(k^2+3)z&=k+9 \end{aligned}$$

(a) Represent this system as an augmented matrix and reduce it to row-echelon form. [3]

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 2 \\ 1 & 2 & k-5 & 2 \\ 3 & 2 & k^2+3 & k+9 \end{array}\right]$$

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 2 \\ -1 & 0 & k-7 & -2 \\ 1 & 0 & k^2+1 & k+5 \end{array}\right] \begin{array}{l} \\ R2-2R1 \\ R3-2R1 \end{array}$$

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 2 \\ -1 & 0 & k-7 & -2 \\ 0 & 0 & k^2+k-6 & k+3 \end{array}\right] \quad R2+R3$$

(b) Determine the value/s of k for which the system will have

(i) no solution [3]

$k^2+k-6=0$

$(k+3)(k-2)=0$

$k=-3, 2$

$k+3 \neq 0$

$k \neq -3$

$\therefore k=2$

(ii) a unique solution [2]

$k^2+k-6 \neq 0$

$k \neq 2, -3$

(iii) infinitely many solutions [2]

$k^2+k-6=0$ $\qquad$ $k+3=0$

$k=2, -3$ $\qquad$ $k=-3$

$\therefore k=-3$

(c) For the value of k obtained in (b)(iii), explain why the system of equations has infinitely many solutions. [2]

THERE IS A COMMON LINE OF INTERSECTION OF THE 3 PLANES.

2. (10 marks)

As part of a stunt in a movie, a car is driven off a cliff 80 metres high at a horizontal speed of 20 m/sec. Assume acceleration due to gravity is 9.8 m/sec^2. Determine each of the following using vector calculus:

(a) the velocity vector and the displacement vector [4]

$t = 0$

$r = 80\mathbf{j}$

$v = 20\mathbf{i}$

$a = -9.8\mathbf{j}$

$v = -9.8t + c$

$20\mathbf{i} = 0 + c$

$v = 20\mathbf{i} - 9.8t\mathbf{j}$

$r = 20t\mathbf{i} - \frac{9.8t^2}{2} + c$

$80\mathbf{j} = 0 - 0 + c$

$r = 20t\mathbf{i} + (80 - 4.9t^2)\mathbf{j}$

(b) when the car hits the ground [2]

$80 - 4.9t^2 = 0$

$t = 4.04$ sec

(c) how far from the edge of the cliff the car hits the ground [1]

$20 \times 4.04 = 80.8$ m

(d) the speed of the car when it hits the ground [2]

$v(4.04) = 20\mathbf{i} - 39.6\mathbf{j}$

SPEED $= 44.4$ m/sec

$\text{Topol}([20, 39.6]) = [44.4 \angle 63.2°]$

(e) the angle of travel at the instant the car hits the ground [1]

$63.2°$

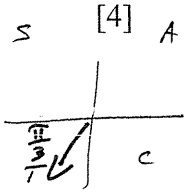
3. (20 marks)

The position of a small body at any time t seconds is given by
$\mathbf{r}(t) = 24\sin\left(\frac{\pi t}{6}\right)\mathbf{i} + 24\cos\left(\frac{\pi t}{6}\right)\mathbf{j}$, $t \geq 0$.

(a) Determine an expression for the velocity $v(t)$ of the body. [2]

$v(t) = 4\pi\cos\frac{\pi t}{6} - 4\pi\sin\frac{\pi t}{6}$

(b) What is the speed of the body when $t = 4$, and what angle to the x-axis is the body moving at this instant? [4]



$v(4) = 4\pi\cos\frac{2\pi}{3} - 4\pi\sin\frac{2\pi}{3}$

$= 4\pi\left(-\frac{1}{2}\right) - 4\pi \times \frac{\sqrt{3}}{2}$

$= -2\pi - 2\sqrt{3}\pi$

$|v(4)| = 4\pi$

ANGLE TO x-AXIS $= \frac{\pi}{3}$

(c) Determine the distance of the body from (0,0) at any time t, and interpret this result in terms of the path described by the body. [3]

$x = 24\sin\frac{\pi t}{6}$ $\quad y = 24\cos\frac{\pi t}{6}$

$\frac{x^2}{24^2} = \sin^2\frac{\pi t}{6}$ $\quad \frac{y^2}{24^2} = \cos^2\frac{\pi t}{6}$

$\sin^2\theta + \cos^2\theta = 1$

$\frac{x^2}{24^2} + \frac{y^2}{24^2} = 1$

$x^2 + y^2 = 24^2$ CIRCULAR

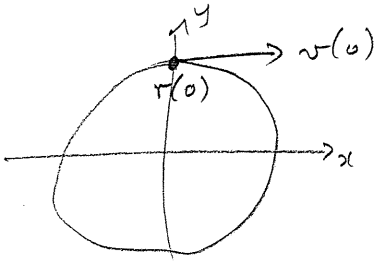
$\therefore$ RADIUS $= 24$ (CENTRE OF CIRCLE (0,0))

(d) Using the result of (c), determine the direction of movement of the body. [2]

$r(0) = 24j$

$v(0) = 4\pi i$

$\therefore$ CLOCKWISE



(e) Determine $r(t) \bullet v(t)$ [2]

$\begin{pmatrix} 24\sin\frac{\pi t}{6} \\ 24\cos\frac{\pi t}{6} \end{pmatrix} \cdot \begin{pmatrix} 4\pi\cos\frac{\pi t}{6} \\ -4\pi\sin\frac{\pi t}{6} \end{pmatrix}$

$= 24\times 4\pi\sin\frac{\pi t}{6}\cos\frac{\pi t}{6} - 24\times 4\pi\sin\frac{\pi t}{6}\cos\frac{\pi t}{6}$

$= 0$

(f) Explain the significance of the answer to (e). [2]

- r PERPENDICULAR TO v
- v TANGENT TO CIRCLE

(g) Determine $\int_0^2 v(t)dt$ and interpret the answer. [3]

$$\left[24\sin\left(\frac{\pi t}{6}\right)i + 24\cos\left(\frac{\pi t}{6}\right)j\right]_0^2$$

$$= \begin{pmatrix} 12\sqrt{3} \\ 12 \end{pmatrix} - \begin{pmatrix} 0 \\ 24 \end{pmatrix}$$

$$= 12\sqrt{3}\,i - 12j$$

DISPLACEMENT VECTOR FROM $r(0)$ TO $r(2)$

(h) Explain why $\int_0^T |v(t)|dt > \int_0^T v(t)dt$ for all $T > 0$. [2]

$\int_0^T |v(t)|\,dt$ IS ARC LENGTH FROM $r(0)$ TO $r(T)$

WHICH IS GREATER THAN CHORD LENGTH

4. (12 marks)

The acceleration of a particle at time t seconds is given by $a(t) = -4i + 2tj$, where distances are measured in centimetres. At $t = 0$ the particle is at the origin and has a velocity $v(t) = 2i + j$

(a) Determine the velocity of the particle when $t = 2$ [2]

$v(t) = -4t\,i + t^2 j + c$

$2i + j = 0 + 0 + c$

$v(t) = (2 - 4t)i + (t^2 + 1)j$

$v(2) = -6i + 5j$

$t = 0$

$r = 0$

$v = 2i + j$

(b) Determine the position of the particle when it is moving parallel to the vertical axis [4]

$r(t) = (2t - 2t^2)i + \left(\frac{t^3}{3} + t\right)j + c$

$0 = 0 + 0 + c$

$r(t) = (2t - 2t^2)i + \left(\frac{t^3}{3} + t\right)j$

$v(t) = (2 - 4t)i + (t^2 + 1)j$

$2 - 4t = 0$

$\frac{1}{2} = t$

$r\left(\frac{1}{2}\right) = 0.5i + \frac{13}{24}j$

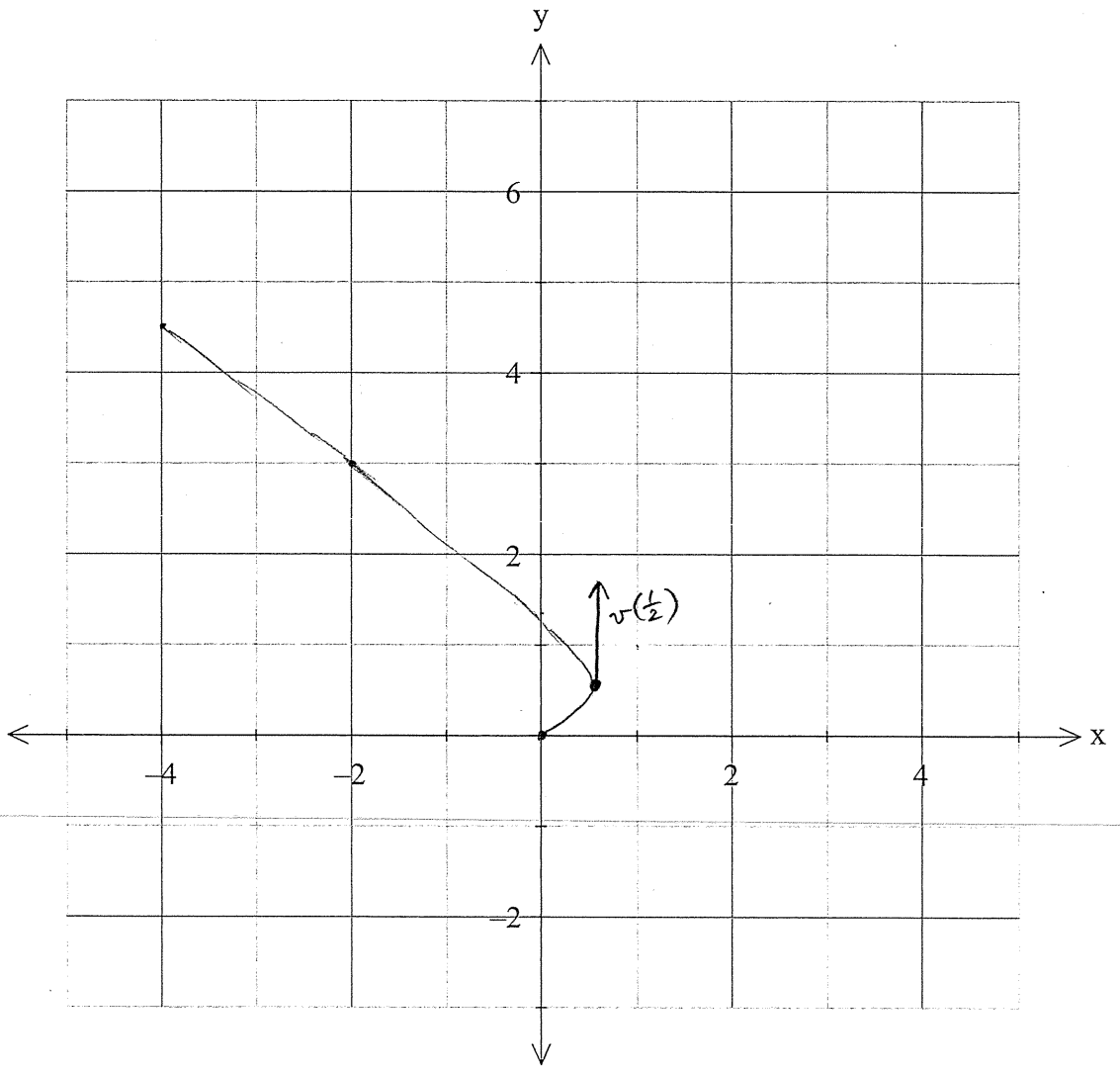
(c) Explain why the particle can never move parallel to the horizontal axis [2]

$t^2 + 1 = 0$

$t^2 = -1$

NO SOLN.

(d) Sketch the path of the particle for $0 \le t \le 2$ [2]



(e) On the axes above sketch $v(0.5)$ [2]

$v(0.5) = 1.25j$